Contraction Coefficients for Noisy Quantum Channels

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Overview

- Review notation and definitions
  - First define using operator convex functions
    a) Relative entropy and generalizations
    b) Riemannian metric or Fisher information
    c) Geodesic distance
  - Basic property — decrease under quantum channels
  - Contraction coefficient: measures how much

- Old Results from Lesniewski/Ruskai

- New Results with F. Hiai
Consider linear ops on $M_d$ space of $d \times d$ matrices as Hilbert space with Hilbert-Schmidt inner prod $\langle P, Q \rangle = \text{Tr} \ P^* Q$

denote adjoint of $\Phi$ by $\hat{\Phi}$, i.e., $\text{Tr} [\Phi(P)]^* Q = \text{Tr} \ P^* \hat{\Phi}(Q)$

Example: Quantum channel $\Phi$

completely positive, trace preserving map on $M_d$, i.e.,

$\mathcal{I} \otimes \Phi$ positivity preserving ("positive") on $M_d \times M_d$

Density matrices $\mathcal{D} \equiv \{P \in M_d : P \geq 0, \text{Tr} \ P = 1\}$

Tangent space $\{A = M_d : A = A^*, \text{Tr} \ A = 0\}$
Basic Tool

Construct operators and functions of operators using

Def. Left and Right mult as linear operators on this vector space

\[ L_P(X) = PX \quad \text{and} \quad R_Q(X) = XQ \]

a) \( L_P \) and \( R_Q \) commute \( L_P[R_Q(X)] = PXQ = R_Q[L_P(X)] \)
b) \( P = P^* \Rightarrow L_P, R_P \) self-adjoint wrt H-S inner prod

For \( P, Q > 0 \) positive definite

c) \( L_P, R_P \) pos def \( \langle X, R_P(X) \rangle = \text{Tr} X^*XP = \text{Tr} XPX^* \geq 0 \)
d) \((L_P)^{-1} = L_{P^{-1}}, \quad (R_Q)^{-1} = R_{Q^{-1}}\)
e) \( f(L_P) = L_{f(P)} \), etc. \( \log R_Q = R_{\log Q} \)
Petz (1986) defined “quasi-entropy”

a.k.a. “generalized relative entropy”, “f-divergence”

\[ G = \{ g : (0, \infty) \mapsto \mathbb{R} \mid \text{operator convex, } g(1) = 0 \} \]

\[ H_g(K, P, Q) \equiv \text{Tr} \sqrt{Q} K^* g_{P}(L P R_{Q}^{-1})(K \sqrt{Q}) \]

**Thm:** \( H_g(K, P, Q) \) jointly convex in \( P, Q \) \( \Rightarrow \) monotonicity results

\[ g(x) \in G \iff \tilde{g}(x) = x g(x^{-1}) \in G \]

\[ \tilde{H}_g(K, P, Q) = H_g(K^*, Q, P) \]

\[ g(x) = x \log x \quad H_g(I, P, Q) = \text{Tr} P (\log P - \log Q) \]

\[ \tilde{g}(x) = - \log x \quad \tilde{H}_g(I, P, Q) = \text{Tr} Q(\log Q - \log P) \]
Recover WYD Entropy

\[ g_t(x) = \begin{cases} \frac{1}{t(1-t)}(x - x^t) & t \neq 1 \\ x \log x & t = 1 \end{cases} \quad t \in (0, 2] \]

\[ \tilde{g}_t(x) = x g_t(x^{-1}) = \begin{cases} \frac{1}{t(1-t)}(1 - x^t) & t \neq 0 \\ - \log x & t = 0 \end{cases} \quad t \in [-1, 1) \]

\[ J_t(K, P, Q) \equiv \text{Tr} \sqrt{Q} K^* g_t(L_P R_Q^{-1})(K \sqrt{Q}) \quad t \in [-1, 2] \]

\[ = \frac{1}{t(1-t)}(\text{Tr} K^* PK - \text{Tr} K^* P^t K Q^{1-t}) \]

\[ J_1(K, P, Q) = \text{Tr} K K^* P \log P - \text{Tr} K^* PK \log Q \]

\[ \tilde{J}_0(K, P, Q) = \text{Tr} K^* K Q \log Q - \text{Tr} K Q K^* \log P \]

Recover both WYD entropy with linear term and \( H(P, Q), \ K = I \)
Riemannian metrics or Fisher information

\[
- \frac{\partial^2}{\partial a \partial b} H_g(P + aA, P + bB, I) \bigg|_{a=b=0} = \text{Tr} A \Omega_P^k(B)
\]

\[
= \langle A, \Omega_P(B) \rangle \equiv \Gamma_P^k(A, B)
\]

for \( \text{Tr} A = \text{Tr} B = 0 \) in LHS, get pos quad form which is RHS with

\[
\Omega_P^k(X) \equiv R_P^{-1} k (L_P R_P^{-1}) X = L_P^{-1} k (R_P L_P^{-1})
\]

with

\[
k(x) = \frac{g(x) + xg(x^{-1})}{(1-x)^2} = \frac{g_{\text{sym}}(x)}{(1-x)^2} \in \mathcal{K}
\]

relate \( H_g \) and \( \Gamma_P^k \)

\[
\mathcal{K} = \{ k : (0, \infty) \mapsto \mathbb{R} \mid k \text{ op convex}, k(x^{-1}) = xk(x) \}
\]

\( \Omega_P \) non-commutative multiplication by \( P^{-1} \)
Examples

\begin{align*}
\text{RelEnt} & \quad k(x) = \frac{\log x}{x-1} \\
\text{WYD} & \quad \Omega_P(X) = \int_0^\infty \frac{1}{P+tl} X \frac{1}{P+tl} \, dt \\
& \quad \frac{1}{t(1-t)} \frac{(1-x^t)(1-x^{1-t})}{(1-x)^2} \\
& \quad t \in [-1, 2] \\
& \quad \frac{1}{2} \left( x^{-t} + x^{-1+t} \right) \\
& \quad t \in [0, 1] \\
\max & \quad \frac{1+x}{2x} \\
\min & \quad \frac{2}{1+x} \\
& \quad \frac{1}{2} \left( X P^{-1} + P^{-1} X \right) \\
& \quad \frac{2}{L_P+R_P}(X)
\end{align*}
Geodesic distance

Define geodesic distance for each \( k \in K \)

\[
D_k(P, Q) \equiv \inf_{\xi(t)} \int_0^1 \sqrt{\text{Tr} \, \xi'(t) \, \Omega_{\xi(t)} \xi'(t)} \, dt
\]

\[
= \inf_{\xi(t)} \int_0^1 \sqrt{\Gamma_{\xi(t)} (\xi'(t) \xi'(t))} \, dt
\]

where \( x(t) \) smooth path with \( \xi(0) = P, \, \xi(1) = Q \)

Know explicitly only in one case, Bures metric

Don’t know if matrix metric \( \| \log P - \log Q \| \) in this framework?

\[
\log P - \log Q = \int_0^\infty \frac{1}{Q + xl} (P - Q) \frac{1}{P + xl} \, dx
\]
Bures metric and trace distance

Know geodesic distance explicitly only for \( \min k(x) = \frac{2}{1+x} \)

\[ \Omega_P(X) = \frac{1}{L_P + R_P}(X) \] Bures metric studied by Uhlmann

\[ D_k(P, Q) = \inf \{ \text{Tr} (Y - Z)^*(Y - Z) : Y^*Y = P, \ Z^*Z = Q \} \]

\[ = 2 \left[ 1 - \text{Tr} \left( \sqrt{PQ} \sqrt{P} \right)^{1/2} \right] = 2 [1 - P \# Q] \]

\( P \# Q \) known as “fidelity” in quantum info

will also use trace distance for which formally

\[ \text{Tr} |P - Q| = H_g(P, Q) \]

with \( g(x) = |x - 1| \) (obviously not op convex since not diff.)
All of above decrease under quantum channels, i.e.,

for any CPT map $\Phi$ and for all $P, Q \in \mathcal{D}$ and $\text{Tr} \ A = 0$

**Thm:** $H_g[\Phi(P), \Phi(Q)] \leq H_g(P, Q) \quad \forall \ g \in \mathcal{G}$

**Thm:** $\text{Tr} \ \Phi(A) \Omega^k_{\Phi(P)} \Phi(A) \leq \text{Tr} \ A \Omega^k_P(A) \quad \forall \ k \in \mathcal{K}$

**Thm:** $D_k[\Phi(P), \Phi(Q)] \leq D_k(P, Q) \quad \forall \ k \in \mathcal{K}$

**Thm:** $|\text{Tr} \ \Phi(P) - \Phi(Q)| \leq \text{Tr} \ |P - Q|$

$$k(x) = \frac{g(x) + xg(x^{-1})}{(1-x)^2} = \frac{g_{\text{sym}}(x)}{(1-x)^2} \text{ op conv and } k(x^{-1}) = xk(x)$$
Definition of Contraction Coefficients

\[ \eta_{g}(\Phi)_{\text{RelEnt}} \equiv \sup_{P, Q} \frac{H_{g}[\Phi(P), \Phi(Q)]}{H_{g}(P, Q)} \]

\[ \eta_{k}(\Phi)_{\text{Riem}} \equiv \sup_{P \in \mathcal{D}, \text{Tr } A = 0} \frac{\text{Tr } \Phi(A) \Omega_{\Phi(P)}^{k} \Phi(A)}{\text{Tr } A \Omega_{P}^{k}(A)} \]

\[ \eta_{k}(\Phi)_{\text{geod}} \equiv \sup_{P, Q} \frac{D_{k}[\Phi(P), \Phi(Q)]}{D_{k}(P, Q)} \]

\[ \eta_{\text{Dob}}(\Phi) = \eta^{\text{Tr}}(\Phi) \equiv \sup_{P, Q} \frac{|\text{Tr } \Phi(P) - \Phi(Q)|}{\text{Tr } |P - Q|} \]

\( \eta^{\text{Tr}} \) quant gen of class Dobrushin coefficient of ergodicity
Old easy results (Lesniewski-Ruskai)

Follow easily from defs with \( k(x) = \frac{g_{\text{sym}}}{(x-1)^2} \)

a) \( \eta_k(\Phi)^{\text{geod}} \leq \eta_k(\Phi)^{\text{Riem}} \leq \eta_{g_{\text{sym}}}^{\text{RelEnt}}(\Phi) \leq \eta_g^{\text{RelEnt}}(\Phi) \leq 1 \)

b) Can also show \( \eta_k^{\text{Riem}}(\Phi) \geq \sqrt{\eta^{\text{Tr}}(\Phi)} \)

c) For unital qubit channels \( \Phi_T : [I + w \cdot \sigma] \mapsto I + (T w) \cdot \sigma \)

\[ \eta_k(\Phi)^{\text{geod}} = \eta_k(\Phi)^{\text{Riem}} = \eta_g^{\text{RelEnt}}(\Phi) \leq 1 = \| T \|^2 \quad \forall \ k, g \]

d) For non-unital CQ qubit channels \( \eta_k(\Phi)^{\text{Riem}} \) depends on \( k \)

Last two stated without proof in Lesniewski-Ruskai
Reformulation of $\eta_k(\Phi)^{\text{Riem}}$ as eigenvalue problem

Use HS inner product $\langle X, Y \rangle = \text{Tr} X^* Y$ and $\hat{\Phi}$ denote adjoint

$$\hat{\Phi} \circ \Omega^k_{\Phi(P)} \circ \Phi(X) = \lambda \Omega^k_P(X)$$

equiv. to

$$[(\Omega_P)^{-1} \circ \hat{\Phi} \circ \Omega^k_{\Phi(P)}] \Phi(X) = \lambda X$$

By max-min principle

$$\lambda_2(\Phi, P) = \sup_{\text{Tr } A = 0} \frac{\langle \Phi(A) \Omega^k_{\Phi(P)} \Phi(A) \rangle}{\langle A \Omega^k_P(A) \rangle}$$

$$\eta^\text{Riem}_k(\Phi) = \sup_P \lambda_2(\Phi, P) = \sup_{P \in \mathcal{D}, \text{Tr } A = 0} \frac{\langle \Phi(A) \Omega^k_{\Phi(P)} \Phi(A) \rangle}{\langle A \Omega^k_P(A) \rangle}$$

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Contraction of Quantum Channels
Recent results and conjectures

Can show \((\Omega_P)^{-1} \circ \hat{\Phi} \circ \Omega^k_{\Phi(P)}\) pos pres \(\Rightarrow\) \(\eta_k^{\text{Riem}}(\Phi) \leq \eta^{\text{Tr}}(\Phi)\)

Cor: \(\eta_k^{\text{Riem}}(x) = x^{-1/2}(\Phi) \leq \eta^{\text{Tr}}(\Phi)\)

Only \(k(x)\) for which both \(\Omega_P\) and \((\Omega_P)^{-1}\) are C.P. is \(x^{-1/2}\) plays important role in quantum Markov processes

Conj: (Kastoryano-Temme) \(\eta_k^{\text{Riem}}(\Phi) \leq \eta_k^{\text{Riem}}(x) = x^{-1/2}(\Phi) \quad \forall \ k \in \mathcal{K}\)

Conj: (Ruskai) \(\eta_k^{\text{Riem}}(\Phi) \leq \eta^{\text{Tr}}(\Phi) \quad \forall \ k \in \mathcal{K}\)

will see both are false
Aside: more on $\mathcal{K}$ and $\Omega_P$

Recall $\mathcal{K} = \{ k : (0, \infty) \mapsto \mathbb{R} \mid k \text{ op convex}, k(x^{-1}) = xk(x) \}$

Can verify $k \in \mathcal{K} \iff \tilde{k}(x) \equiv 1/k(x^{-1}) \in \mathcal{K}$

$\mathcal{K}$ is a convex set with extreme points $k_\nu = \frac{1+x}{(x+\nu)(1+x\nu)} \frac{(1+\nu)^2}{2}$

$$k(x) = \int_0^1 \left( \frac{1}{x+\nu} + \frac{1}{1+x\nu} \right) \frac{1+\nu}{2} dm(\nu) = \int_0^1 \frac{1+x}{(x+\nu)(1+x\nu)} \frac{(1+\nu)^2}{2} dm(\nu)$$

$$\Omega_P^k(X) \equiv R_P^{-1} k \left( L_P R_P^{-1} \right) X = L_P^{-1} k \left( R_P L_P^{-1} \right)$$

Petz uses $f = 1/k$ with different conventions but equivalent result

$\Omega_P$ non-commutative multiplication by $P^{-1}$

$(\Omega_P)^{-1} \neq \Omega_{P^{-1}}$ non-commutative multiplication by $P$
Aside on CP of $\Omega_P$ and $(\Omega_P)^{-1}$

Example: $\Omega_P^{\log}(X) = \int_0^\infty \frac{1}{P + tl} X \frac{1}{P + tl} dt$

$$(\Omega_P^{\log})^{-1}(Y) = \int_0^\infty P^u Y P^{1-u} du$$

$\mathcal{K} = \{ k : (0, \infty) \mapsto \mathbf{R} \mid k \text{ op convex}, k(x^{-1}) = xk(x) \}$

$\mathcal{K}^+ = \{ k \in \mathcal{K} : \Omega_P^k \text{ is C.P.} \ \forall \ P \in \mathcal{D} \}$

$\mathcal{K}^- = \{ k \in \mathcal{K} : (\Omega_P^k)^{-1} \text{ is C.P.} \ \forall \ P \in \mathcal{D} \}$

$k(x) \in \mathcal{K}^+ \iff \tilde{k}(x) = 1/k(x^{-1}) \in \mathcal{K}^-$

$\mathcal{K}^+ \cap \mathcal{K}^- = \{ k(x) = x^{-1/2} \}$

WYD $k_t(x) \in \mathcal{K}^+$ if $t \in [0, 1]$, $\in \mathcal{K}^-$ if $t \in [-1, -\frac{1}{2}] \cup (\frac{3}{2}, 2]$
Def: $k_1 \preceq k_2$ if $k_1(e^t)/k_2(e^t)$ is pos def in Bochner sense

Fourier transform is positive

Equiv cond: matrix with els $\frac{k_1(x_j/x_k)}{k_2(x_j/x_k)}$ is pos semi-def

Thm: TFAE

a) $k \in \mathcal{K}^+$, i.e., $\Omega_P$ is C.P.

b) $k \preceq x^{-1/2}$

c) $F(t) = e^t k(e^{2t})$ is pos def

Get above results and analyze many families using these conds
Figure: Diagram of families in $\mathcal{K}$ parameterized to increase in $\precsim$ order with the lower ball for $\mathcal{K}^+$ and the upper $\mathcal{K}^-$. The red curve describes the Heinz family $k^H_\alpha (0, \frac{1}{2})$ and $\tilde{k}^H_\alpha (\frac{1}{2}, 1)$; the blue curve the binomial family $k^B_{-\alpha} (-1, 1)$; the green curve the power difference family $k^\text{PD}_{-\alpha} (-2, 1)$. The left brown curve $k^\text{WYD}_t$ with $t \in [\frac{1}{2}, 2]$ and the right dotted brown curve the dual $\tilde{k}^\text{WYD}_t$. Note crossings at $\frac{4}{(1+\sqrt{x})^2}$ and $\frac{\log x}{x-1}$.
Hiai’s new results

**Thm:** (Hiai-Petz) $\langle A, \Omega_p^k(A) \rangle = \lim_{\epsilon \to 0} D_k(P, P + \epsilon A)$

**Cor:** $\eta^\text{Riem}_k(\Phi) \leq \eta^\text{geod}_k(\Phi)$ $\Rightarrow$ **Cor:** $\eta^\text{Riem}_k(\Phi) = \eta^\text{geod}_k(\Phi)$

since opposite ineq elementary

**Thm:** (Hiai) $\eta^\text{RelEnt}_\log(\Phi) = \eta^\text{Riem}_\log(\Phi)$

**Thm:** (Hiai) $\eta^\text{Riem}_\text{Bur}(\Phi) = \eta^\text{RelEnt}_\text{Bur}(\Phi)$ $k(x) = \frac{2}{1+x}$

**Conj:** $\eta^\text{RelEnt}_g(\Phi) = \eta^\text{Riem}_k(\Phi)$ $\forall$ $k(x) = \frac{g_{\text{sym}}}{(x-1)^2}$
Unital qubit channels

\[ \Phi : P = \frac{1}{2} [I + w \cdot \sigma] \mapsto \frac{1}{2} [I + \sum_k \alpha_k w_k \sigma_k] \]

rep in Pauli basis \( \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \) = \( \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_3 \end{pmatrix} \)

\[ \eta_k(\Phi)^{\text{geod}} = \eta_k(\Phi)^{\text{Riem}} = \eta_g^{\text{RelEnt}}(\Phi) \leq 1 = \|T\|^2 = \max_k \alpha_k^2 \quad \forall k, g \]

Proof exploits fact that for \( \Gamma_w \) pos lin op on \( \mathbb{R}^3 \)

\[ \sup_{y \in \mathbb{R}^3} \frac{\langle Ty, \Gamma^{-1}_{T Tw} Ty \rangle}{\langle y, \Gamma^{-1}_{w} y \rangle} = \sup_{y \in \mathbb{R}^3} \frac{\langle T^* y, \Gamma_w T^* y \rangle}{\langle y, \Gamma_{Tw} y \rangle}. \]
CQ qubit channels

$$\Phi : P = \frac{1}{2}[I + w \cdot \sigma] \mapsto \frac{1}{2}[I + \alpha w_1 \sigma_x + \tau \sigma_z]$$

CP cond  \(\alpha^2 + \tau^2 \leq 1\)

extreme \(k_{\nu} = \frac{1+x}{(x+\nu)(1+x\nu)} \frac{(1+\nu)^2}{2}\)

\(\eta_{k_{\nu}}^{\text{Riem}} = \frac{\alpha^2}{1-\left(\frac{1-\nu}{1+\nu}\right)^2 \tau^2} \geq \alpha^2\)

implies \(\eta_{k}^{\text{Riem}}(\Phi_{\alpha,\tau})\) depends on \(k(x)\) and \(\geq \alpha^2 \ \forall \ k \in \mathcal{K}\)

Also since \(\eta_{k}^{\text{Riem}}(\Phi) \geq \sqrt{\text{Tr}^\Phi}(\Phi)\) we have

\(\eta^\text{Tr}(\Phi_{\alpha,\tau}) = \alpha \implies \eta_{k}^{\text{Riem}}(\Phi_{\alpha,\tau}) \geq \alpha^2 \ \forall \ k \in \mathcal{K}\)
Can show by direct (usually tedious for \(=\)) computation

<table>
<thead>
<tr>
<th>harm</th>
<th>(\widetilde{WY})</th>
<th>geom</th>
<th>log</th>
<th>(WY)</th>
<th>Bur</th>
</tr>
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<tbody>
<tr>
<td>(k(x))</td>
<td>(\frac{1+x}{2x})</td>
<td>(\frac{(1+\sqrt{x})^2}{4x})</td>
<td>(x^{-1/2})</td>
<td>(\log x)</td>
<td>(\frac{4}{x-1})</td>
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\[=\quad \geq \quad \geq \quad \geq \quad = \quad =\]

\(\eta^\text{Riem}_k\) | \(\frac{\alpha^2}{1-\tau^2}\) | \(\frac{1+\sqrt{1-\tau^2}}{2(1-\tau^2)}\) | \(\frac{\alpha^2}{\sqrt{1-\tau^2}}\) | \(\frac{\alpha^2}{2\tau}\) | \(\log \frac{1+\tau}{1-\tau}\) | \(\frac{2\alpha^2}{1+\sqrt{1-\tau^2}}\) | \(\alpha^2\)

Can verify both \(k(x)\) and \(\eta^\text{Riem}_k\) in decreasing order above

**Conj:** Equality holds, at least when \(\alpha^2 + \tau^2 = 1\) (not easier)

some numerical evidence against
Show some conjectures false

\[ k(x) = \frac{1+x}{2x}, \quad \frac{(1+\sqrt{x})^2}{4x}, \quad \frac{x^{-1/2}}{\log x \over x-1} \quad \frac{4}{(1+\sqrt{x})^2} \quad \frac{2}{1+x} \]

\[ \eta^\text{Riem}_k = \frac{\alpha^2}{1-\tau^2}, \quad \frac{1+\sqrt{1-\tau^2}}{2(1-\tau^2)}, \quad \frac{\alpha^2}{\sqrt{1-\tau^2}}, \quad \frac{\alpha^2}{2\tau} \log \frac{1+\tau}{1-\tau} \quad \frac{2 \alpha^2}{1+\sqrt{1-\tau^2}} \quad \alpha^2 \]

- For \( \frac{1+x}{2x} \) and \( \alpha < 1 - \tau^2 \), \( \eta^\text{Riem} = \frac{\alpha^2}{1-\tau^2} > \alpha = \eta^\text{Tr} \) for \( \tau \neq 0 \)
  
  e.g., for \( \alpha = \tau = 1/\sqrt{2} \), \( \eta^\text{Riem}_{\frac{1+x}{2x}} = 1 > \alpha = \eta^\text{Tr} \)

- Also shows can have some = 1 and others < 1

- For \( x^{-1/2} \) with \( \alpha^2 + \tau^2 = 1 \), \( \eta^\text{Riem} = \eta^\text{Tr} = \alpha \) not largest

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Contraction of Quantum Channels
Further calculations show

- Can have $\eta_g^{\text{RelEnt}} > \eta_k^{\text{Riem}}$ for ext points.

**Summary:** In general with strict $<$ possible (probably generic)

$$\sqrt{\eta^{\text{Tr}}(\Phi)} \leq \eta_k(\Phi)^{\text{geod}} = \eta_k(\Phi)^{\text{Riem}} \leq \eta_{g_{\text{sym}}}(\Phi) \leq \eta_g^{\text{RelEnt}}(\Phi) \leq 1$$
Open questions

Still some open questions about when $\Omega^k_P$ C.P.

Equality in some $\eta^\text{Riem}_k$ in bounds for CQ channel?

$\eta$ for unital channels with $d > 2$??

$\eta$ for random unitaries??

When does $k_m$ inc p.w or in $\leq$ order $\Rightarrow \eta^\text{Riem}_{km}(\Phi)$ increase?

or decrease? Hiai partial results for QC and CQ channels

When does some $\eta_{km}(\Phi) = 1 \Rightarrow$ others $= 1$ also ?

When does some $\eta_{km}(\Phi) < 1 \Rightarrow$ others $< 1$ also ?

How to number slides backwards in beamer??
References

D. Petz (1986) and (1996) and others given in refs below

A. Lesniewski and M. B. Ruskai, “Monotone Riemannian metrics and relative entropy on noncommutative probability spaces”  

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